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Fixed Point Theorems in Fuzzy Uniform Space

Abstract

In this paper, some common fixed point theorems are proved for pair of weakly compatible mappings and semi compatible mappings for implicit relation in uniform space with the notation of E-distance.

Keywords: Fuzzy Uniform Spaces, Common Fixed Point, E & A-Distance, Contractive Maps, Implicit Relation, Weakly Compatible Maps.

Mathematical Classification (2000)

49H10, 54H25.

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Introduction

In 1965, Zadeh [7] worked on theory of fuzzy set, which provided important contribution in the field of pure and applied mathematics.

Review of Literature

Kramosel and Michalek [5], have introduced the concept of Fuzzy Metric Space. George and Veeramani [3], introduced the Housedorff topology on fuzzy metric space. Cho. Et. al. [2] introduced the notion of semi compatible maps in a d- topological space and Junjck & Rhoades [4] termed a pair of self maps to be coincidentally commuting or equivalently weak compatible if they commute at their coincidence points. With the help of these A-distance and E-distance we prove common fixed point theorems for weak compatible mappings and semi compatible mappings with implicit relation. Recently in 2005 the concept of A-distance and E-distance in uniform space was introduced by Aamri and Moutawakil [1]. By using all the concepts mentioned above we have introduced fuzzy Uniform space and proved a fixed point result in this present paper.

We call fuzzy uniform space (X, ϑ_α, t) a non empty set X endowed of an fuzzy uniformity $\vartheta_\alpha(t)$, the latter being a special kind of filter on $X \times X \to [0,\infty)$ with $\alpha \in [0,1]$ all whose elements contain the diagonal $\Delta\alpha(t) = \{(x,x,t)\alpha / x \in X\}$. If $V_\alpha(t) \in \vartheta_\alpha(t)$ and $(x,y,t)_\alpha \in V_\alpha(t), (y,x,t)_\alpha \in V_\alpha(t)$, x and y are said to be $V_\alpha(t)$-close and a sequence (x^n) in X is a Cauchy sequence for $\vartheta_\alpha(t)$ if for any $V_\alpha(t) \in \vartheta_\alpha(t)$, there exists $N \geq 1$ such that $x^n$and x^m are $V_\alpha(t)$-close for $n, m \geq N$. An fuzzy uniformly $\vartheta_\alpha(t)$ defines a unique topology $T(\vartheta_\alpha(t))$ on X for which the neighborhoods of $x \in X$ are the sets $V_\alpha(t)(x) = \{y \in X / (x,y,t)_\alpha \in V\alpha t$ when $V\alpha t$ runs over $\vartheta\alpha t$. A fuzzy uniform space $X, \vartheta\alpha t$ is said to be Hausdorff if and only if the intersection of all the $V_\alpha \in \vartheta_\alpha$ reduces to the diagonal Δ_α of X i.e., if $(x,y,)_\alpha \in V_\alpha$ for all $V_\alpha(t) \in \vartheta_\alpha(t)$ implies $x = y$ with respect to t.

This guarantees the uniqueness of limits of sequences.$V_\alpha(t) \in \vartheta_\alpha(t)$ is said to be symmetrical if $V_\alpha(t) = V^{-1}{}_\alpha(t) = \{(y,x,t)_\alpha / (x,y,t)_\alpha \in V_\alpha(t)\}$. Since each $V_\alpha(t) \in \vartheta_\alpha(t)$ contains a symmetrical $W_\alpha(t) \in \vartheta_\alpha(t)$ and if $(x,y,t)_\alpha \in W_\alpha(t)$ then x and y are both $W_\alpha(t)$ and $V_\alpha(t)$–close with respect to t, then for our purpose, we assume that each $V_\alpha(t) \in \vartheta_\alpha(t)$ is symmetrical. When topological concepts are mentioned in the context of a fuzzy uniform space $(X, \vartheta_\alpha(t))$ they always refer to the topological space$(X, T(\vartheta_\alpha(t))$.

Popa in [6] used the family Φ of implicit function to find the fixed points of two pairs of semi compatible maps in a d complete topological space, where Φ be the family of real continuous function $\emptyset: (R^+)^4 \to R$ satisfying the properties

(F_h) for every $u \geq 0, v \geq 0$ with $\phi(u,v,u,v) \leq 0$ or $\phi(u,v,v,u) \leq 0$ we have $u \leq hv$, $0 < h < 1$.

(F_u) $\phi(u,,0,0,u) \leq 0$ implies that $u \leq 0$.

We also involve a non-decreasing function ψ on R^+ such that $\psi(t) < t$, for $t > 0$ and $\psi^n(t) = 0$ as $n \to \infty$.

Aim of the Study

Our aim in this present paper, to prove some common fixed point theorems for pair of weakly compatible mappings and semi compatible mappings for implicit relation but in uniform space with the notation of E-distance, which will generalized almost all the above mentioned/referred results.

Preliminaries

Definition 2.1

[1] Let $(X, \vartheta_\alpha, (t))$ be a Fuzzy uniform space. A function $p: X \times X \to [0, \infty) \to R^+$ is said to be an A-distance if for any $V_\alpha \in \vartheta_\alpha$ there exists $\delta > 0$ such that if $p(z, x, t)\,\alpha \le \delta$ and $p(z, y, t)\,\alpha \le \delta$ for some $z \in X$, then $(x, y, t)_\alpha \in V_\alpha(t)$

Definition 2.2 [1]

Let $(X, \vartheta_\alpha, (t))$ be fuzzy uniform space. A function $p: X \times X \to [0, \infty) \to R^+$ is said to be an E-distance if

(p_1) p is an A-distance ,

(p_2) $p(x, y, t)_\alpha \le p(x, z, t)_{\alpha 1} + p(z, y, t)_{\alpha 2} \quad \forall\, x, y, z \in X$ and $\alpha_1 + \alpha_2 = \alpha$.

Definition 2.3

Let$(X, \vartheta_\alpha, (t))$ be uniform space and p be an A- distance on X.

1. X is complete if for every p-Cauchy sequence $\{x_n\}$, there exists x in X such that $\lim_{n\to\infty} p(x_n, x, t) = 0$.
2. X is p-Cauchy complete if for every p-Cauchy sequence $\{x_n\}$, there exists x in X such that, $\lim_{n\to\infty} x_n = x$ with respect to $T(\vartheta_\alpha(t))$

(III) $f : X \to X$ is p-continuous if $\lim_{n\to\infty} p(x_n, x, t) = 0$ implies$\lim_{n\to\infty} p(f(x_n), f(x), t)_\alpha = 0$. (IV) $f : X \to X$ is $T(\vartheta_\alpha(t))$-continuous if $\lim_{n\to\infty} x_n = x$ with respect to $T(\vartheta_\alpha(t))$ implies

$\lim_{n\to\infty} f(x_n) = f(x)$ with respect to $T(\vartheta_\alpha(t))$.

(V) X is said to be p-bounded if$\delta_p(X)_\alpha = sup\left\{p\left(x, y_\alpha(t)\right)/x, y \in X\right\} < \infty$.

Definition 2.4: [2]

Let $(X, \vartheta_{\alpha,}(t))$ be fuzzy uniform space and p be an E-distance on X . Two self maps S and T on are said to be semi compatible

If $\lim_{n\to\infty} p(STx_n, Tx, t)_\alpha = 0$, whenever $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} ST = X$.

Lemma 2.1: [3]

Let $(X, \vartheta_\alpha, (t))$ be a Hausdourff uniform space and p be an A-distance on X .

Let $\{x_n\}, \{y_n\}$ be arbitrary sequences in X and $\{\alpha_n\}, \{\beta_n\}$ be sequences in R^+ and converging to 0. Then , for $x, y, z \in X$, the following holds

(a) If $p(x_n, y, t)_\alpha \le \alpha_n$ and $p(x_n, z, t)_\alpha \le \beta_n$ for all $n \in N$,then $y = z$,
In particular, if $p(x, y, t)_\alpha = 0$ and $p(x, z, t)_\alpha = 0$, then $y = z$.

(b) If $p(x_n, y_n, t)_\alpha \le \alpha_n$ and $p(x_n, z, t)_\alpha \le \beta_n$ for all $n \in N$,then $\{y_n\}$ converges to z.

(c) If $p(x_n, x_m, t)_\alpha \le \alpha_n$for all $m > n$, then $\{x_n\}$ is a Cauchy sequence in$\left(X, \vartheta_{\alpha,}(t)\right)$.

Main Result

Theorem 3.1

Let $\left(X, \vartheta_{\alpha,}(t)\right)$ be a Hausdorff uniform space and p be an E-distance on p-cauchy complete space$X$. Let A, B, S and T be self mappings of X satisfying that

(I) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$;

(II) the pair (A, S) is semi compatible and (B, T) is weak compatibility;

(I) S is continuous;

(II) $\emptyset(p(Ax, By, t)_\alpha, p(Sx, Ty, t)_\alpha, p(Ax, Sx, t)_\alpha, p(Sx, by, t)_\alpha, p(By, Ty, t)_\alpha, p(Ax, Ty, t)_\alpha, \le 0$.
for some $\phi \in \Phi$ and $\forall\, x, y \in X$.

Then A, B, S and T have unique common fixed point.

Proof

Let x_0 be any point in X. As $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, there exist x_1 and $x_2$in X such that $A(x_0) = T(x_1)$ and $B(x_1) = S(x_2)$. In general we can construct sequence $\{y_n\}$ in X such that

$y_{2n+1} = A(x_{2n}) = T(x_{2n+1})$ and$y_{2n+2} = B(x_{2n+1}) = S(x_{2n+2})$, $n = 0,1,2, \ldots\ldots$

Now by (IV) $p(y_{2n+1}, y_{2n+2}, t)_\alpha = P(A(x_{2n}), B(x_{2n+1}), \mathrm{t})_\alpha$

$$\emptyset(P(A(x_{2n}), B(x_{2n+1}), \mathrm{t})_\alpha, p\left(Sx_{2n,}Tx_{2n+1}\right)_\alpha, p\left(Ax_{2n,}Sx_{2n}\right)_\alpha, p\left(Sx_{2n,}Bx_{2n+1}, t\right)_\alpha,$$

$$p\left(Bx_{2n+1,}Tx_{2n+1}, t\right)_\alpha, p\left(Ax_{2n,}Tx_{2n+1}, T\right)_\alpha) \le 0$$

$$\Rightarrow \emptyset(p(y_{2n+1}, y_{2n+2}, t)_\alpha, p\left(y_{2n,}y_{2n+1}, t\right)_\alpha, p\left(y_{2n+1,}y_{2n}, t\right)_\alpha, p\left(y_{2n,}y_{2n+2}, t\right)_\alpha,$$

$$p\left(y_{2n+2,}y_{2n+1}, t\right)_\alpha, p\left(y_{2n+1,}y_{2n+1}, t\right)_\alpha) \le 0$$

By (F_h)

$$p(y_{2n+2}, y_{2n+1}, t)_\alpha \le p(y_{2n+1}, y_{2n}, t)_\alpha$$

$$\Rightarrow p(y_{2n+2}, y_{2n+1}, t)_\alpha \le p(y_{2n+1}, y_{2n}, t)_\alpha$$

Again putting $x = x_{2n+2}$ and $y = y_{2n+1}$ in (IV), we have

$$\emptyset(p(y_{2n+3}, y_{2n+2}, t)_\alpha, p\left(y_{2n+2,}y_{2n+1}, t\right)_\alpha, p\left(y_{2n+3,}y_{2n+2}, t\right)_\alpha, p\left(y_{2n+1,}y_{2n+3}, t\right)_\alpha,$$

$$p\left(y_{2n+1,}y_{2n+2}, t\right)_\alpha, p\left(y_{2n+3,}y_{2n+1}, t\right)_\alpha) \le 0$$

By (F_h)

$$p(y_{2n+3}, y_{2n+2}, t)_\alpha \le p(y_{2n+2}, y_{2n+1}, t)_\alpha$$

Therefore in general

$$p(y_n, y_{n+1}, t)_\alpha \leq h[p(y_{n-1}, y_{n+1}, t)_\alpha].$$

$$\Rightarrow p(y_n, y_{n+p}, t)_\alpha \leq h^n[\delta_p(X)(t)] \text{ where } \delta_p(X)_\alpha(t) = sup\{p(x, y, t)_\alpha / x, y \in X\}.$$

Then by ψ_2 and lemma $1(c)$ we have $\{y_n\}$is cauchy sequence in X and X is S^* complete therefore $\lim_{n\to\infty} p(y_n, z) = 0.$ By lemma *1 (b)* there exist sequences αn and βn

which converging to 0. There the subsequences $A(x_{2n})$, $T(x_{2n+1})$, $B(x_{2n+1})$ and $S(x_{2n+2})$ also converge to z. Since S is continuous then $SA(x_{2n}) \to Sz$ and $SS(x_{2n}) \to Sz$. By the semi compatibility of the pair (A, S) gives $AS(x_{2n}) \to Sz$ as $n \to \infty$.

By (IV)

$$\emptyset\begin{bmatrix} p(AS(x_{2n}), B(x_{2n+1}), t)_\alpha, p(SS(x_{2n}), T(x_{2n+1}), t)_\alpha, p(AS(x_{2n}), SS(x_{2n}), t)_\alpha, \\ p(SS(x_{2n}), T(x_{2n+1}), t)_\alpha, p(T(x_{2n+1}), B(x_{2n+1}), t)_\alpha, p(AS(x_{2n}), T(x_{2n+1}), t)_\alpha \end{bmatrix}$$

Letting $n \to \infty$, we get

$$\emptyset[p(Sz, z, t)_\alpha, p(Sz, z, t)_\alpha, p(Sz, z, t)_\alpha, p(Sz, z, t)_\alpha, p(z, z, t)_\alpha] < hp(Sz, z, t)_\alpha.$$

$\Rightarrow p(Sz, z, t)_\alpha = 0.$

Again put $x = z$ and $y = x_{2n+1}$ in (IV)

$$\leq \emptyset\begin{bmatrix} p(A(z), B(x_{2n+1}), t)_\alpha, p(S(z), T(x_{2n+1}), t)_\alpha, p(Az, Sz, t)_\alpha, p(S(z), B(x_{2n+1}), t)_\alpha, \\ p(T(x_{2n+1}), B(x_{2n+1}), t)_\alpha, p(A(z), T(x_{2n+1}), t)_\alpha \end{bmatrix}$$

$$\leq \emptyset[p(Az, z, t)_\alpha, p(Sz, z, t)_\alpha, p(Az, Sz, t)_\alpha, p(Sz, z, t)_\alpha, p(z, z, t)_\alpha, p(Az, z, t)_\alpha]$$

$\leq [p(Az, z, t)_\alpha, 0, p(Az, Sz, t)_\alpha, p(Sz, z, t)_\alpha, 0, p(z, z, t)_\alpha, p(Az, z, t)_\alpha] < p(Az, z, t)_\alpha$

$\Rightarrow p(Az, z, t)_\alpha = 0.$

Now

$p(Sz, z, t)_\alpha = 0.$ and $p(Az, z, t)_\alpha = 0.$

Hence $Sz = Az.$

$p(z, z, t)_\alpha = p(z, Az, t)_\alpha = p(Az, z, t)_\alpha.$

$\Rightarrow p(z, z, t)_\alpha = 0$ and $p(Sz, z, t)_\alpha = 0.$ Hence $z = Sz = Az.$

Case I. Since $A(X) \subseteq T(X)$, therefore there exists u in X such that $Az = Tu$.

Put $x = x_{2n}$ and $y = u$ in (IV)

$$\emptyset\begin{bmatrix} p(A(x_{2n}), B(u), t)_\alpha, p(S(x_{2n}), T(u), t)_\alpha, p(A(x_{2n}), S(x_{2n}), t)_\alpha, p(S(x_{2n}), B(u), t)_\alpha, \\ p(T(u), B(u), t)_\alpha, p(A(x_{2n}), T(u), t)_\alpha \end{bmatrix}$$

$\emptyset[p(z, Bu, t)_\alpha, p(z, Az, t)_\alpha, p(z, z, t)_\alpha, p(z, Bu, t)_\alpha, p(z, Bu, t)_\alpha, p(z, Az, t)_\alpha] < hp(z, Bu, t)_\alpha$

$\Rightarrow p(z, Bu, t)_\alpha = 0.$

And so, $p(Az, z, t)_\alpha = 0.$

i.e. $p(T(u), z, t)_\alpha = 0$

$\Rightarrow Tu = Bu.$

By the weak compatibility of the pair (B, T) $BTu = TBu \Rightarrow Bz = Tz.$

Case II. Put x = z and y = z in (IV), we get

$$\leq \emptyset\begin{bmatrix} p(A(z), B(z), t)_\alpha, p(S(z), T(z), t)_\alpha, p(S(z), A(z), t)_\alpha, p(S(z), B(z), t)_\alpha, \\ p(T(z), B(z), t)_\alpha, p(A(z), T(z), t)_\alpha \end{bmatrix}$$

$$\leq \emptyset[p(z, B(z)), p(z, T(z)), p(z, z), p(z, B(z)), p(Tz, Bz), p(z, T(z))] < p(z, B(z)).$$

$\Rightarrow p(z, B(z)) = 0$ and already $p(z, z) = 0.$ $Bz = z.$ Hence $Az = Bz = Sz = Tz.$

Uniqueness: Let z_1 and z_2 are two common fixed point of A, S B and T. Then

$Az_1 = Bz_1 = Sz_1 = Tz_1$ and $Az_2 = Bz_2 = Sz_2 = Tz_2$.

$$\leq \emptyset\begin{bmatrix} p(A(z_1), B(z_2), t)_\alpha, p(S(z_1), T(z_2), t)_\alpha, p(S(z_1), A(z_1), t)_\alpha, p(S(z_1), B(z_2), t)_\alpha, \\ p(T(z_2), B(z_2), t)_\alpha, p(A(z_1), T(z_2), t)_\alpha \end{bmatrix}$$

$$\leq \emptyset[p(z_1, z_2, t)_\alpha, p(z_1, z_2, t)_\alpha, p(z_1, z_1, t)_\alpha, p(z_2, z_2, t)_\alpha, p(z_2, z_1, t)_\alpha, p(z_1, z_2, t)_\alpha]$$

$< hp(z_1, z_2, t)_\alpha$

$p(z_1, z_2, t)_\alpha = 0.$

Again $p(z_1, z_1, t)_\alpha \leq p(z_1, z_2, t)_\alpha + p(z_2, z_1, t)_\alpha$

Hence $z_1 = z_2$. that is uniqueness proved.

Conclusion

Theorems proved above are generalizing the results of [3], [2],[[6] and [1], which fulfill our aim.

Endnotes

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